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THE "VISCO-PLASTIC" APPROXIMATION TO HART'S CONSTITUTIVE LAW FOR INELASTIC DEFORMATION

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Abstract—Hart's constitutive law for inelastic deformation of metals in uniaxial loading is described. For sufficiently low temperature or high strain rates, Hart's constitutive law in uniaxial loading is accurately approximated by a simpler law termed the visco-plastic approximation. In the visco-plastic approximation Hart's "plastic" element reduces to a classical rate independent power-law work hardening plastic material. An error bound for the visco-plastic approximation is estimated. Analyses are carried out for some simple deformation histories, including a constant plastic strain rate test, to elucidate the content of Hart's law.

NOTATION

a	"anelastic" strain, [dimensionless]
$\dot{a}^*$	constant in non-linear dashpot description, $[\text{time}]^{-1}$
b	slope of y vs x trajectory in constant $\dot{\varepsilon}$ test
B	contribution of Γ properties to $\Delta\alpha$, [dimensionless]
C	constant in equation of plastic state, [dimensionless]
D	constant in equation of plastic state, $[\text{time}]^{-1}$
E	Young's modulus, [stress]
f	constant used to determine D, $[\text{time}]^{-1}$
$f(x, y)$	dy/dx in a constant $\dot{\varepsilon}$ test
$F(\lambda)$	a function of λ with value close to 1
G	elastic shear modulus, [stress]
$g(x, b)$	isocline with slope b in const. $\dot{\varepsilon}$ test
$h(x)$	hardness coefficient Γ evaluated at $y = x$, [dimensionless]
$h_1(x)$	derivative of Γ with respect to y at $y = x$, [dimensionless]
k	constant in Hart's hardening function Γ, [dimensionless]
K_m	effective testing machine stiffness, [stress]
K_T	effective combined sample and machine stiffness, [stress]
m	constant used for evaluating Γ, [dimensionless]
M	power law exponent for the frictional dashpot, [dimensionless]
$\mathscr{M}$	the "anelastic" modulus, [stress]
$P(x)$	1 if γ in Ω_2 (Fig. 4), 0 otherwise
Q	"activation energy for self diffusion", [cal g^{-1} mol^{-1}]
R	gas constant = 1.986 cal (g^{-1} mol^{-1} K^{-1})
T	absolute temperature
t	time
t_T	time at the end of load history
x	dimensionless $\sigma^* = \sigma^*/G$
x_T	x at end of load history
x_{max}	the value of x on γ_0 corresponding to y_{max}
y	dimensionless $\sigma_a = \sigma_a/G$
y_T	value of y at end of load history
y_{max}	the maximum value of y in the load history
α	"plastic" strain = strain in $\dot{\alpha}$ element, [dimensionless]
$\dot{\alpha}_0$	a particular value of $\dot{\alpha}$, $[\text{time}]^{-1}$
$(\dot{\alpha}_0)_{min}$	the value of $\dot{\alpha}_0$ which gives the best $\Delta\alpha$
α_1	accumulated α while γ in Ω_1
α_2	accumulated α while γ in Ω_2
α_{vp}	accumulated α predicted by visco-plastic approximation
α_{vp0}	α_{vp} if initial σ^* is 0
β	$1 + (k/m)[1 - (x_0/y_{max})^m]$
δ	$[(D/\dot{\alpha})x^m]^\lambda$, a useful parameter, [dimensionless]
δ_0	$[(D/\dot{\alpha}_0)(x_{max})^m]^\lambda$
$(\delta_0)_{min}$	δ_0 used to minimize $\Delta\alpha$
δ_1	$D/\dot{\varepsilon}$, a small parameter (not related to δ)
δ_2	$DC/\mathscr{M}\dot{\varepsilon}$, a small parameter (not related to δ)
$\Delta\alpha$	error bound for α due to visco-plastic approximation
$(\Delta\alpha)_{min}$	tightest bound on $\Delta\alpha$
ε	inelastic strain = $a + \alpha$, [dimensionless]

ε_e elastic strain
ε_t total strain $= \varepsilon_e + \varepsilon = \varepsilon_e + \alpha + a$
$\dot{\varepsilon}^*$ $D(\sigma^*/G)^m$, [time]$^{-1}$
η dummy variable for integration
γ curve defined by $x(t)$, $y(t)$
γ_0 curve $x(t)$, $y(t)$ when $\dot{\alpha} = \dot{\alpha}_0$
Γ hardening coefficient for "plastic" state, [dimensionless]
λ constant used in plastic state relation $\simeq 0.15$
Ω accessible region on x, y plane for a given y_{max} and x_T
Ω_1 region of Ω below γ_0
Ω_2 region of Ω above γ_0
σ total true stress
σ_a "anelastic" stress carried by plastic branch in Hart's model
σ_f stress carried by "frictional" element in Hart's model
σ^* "hardness" = internal state variable in Hart's model, [stress].

INTRODUCTION

Hart and co-workers[1–3] have proposed a constitutive relation for inelastic deformation of solids. This law, based on an earlier simpler law[4], relates the stress to the deformation history using state variables which evolve during deformation.

One motivation for the construction of such constitutive laws is to accurately characterize material behavior for use in structural and metal forming computations. Since the single constitutive law is meant to encompass transient effects due to various deformation mechanisms it must be fairly complex. If it fulfills its purpose, however, it should reduce to the common simpler descriptions such as elasticity, rate-independent plasticity, and power-law creep to the same extent that real material response reduces to these descriptions. With this view, the constitutive law itself may be viewed as a physical material subject to investigation. One direct way of gaining understanding of the constitutive law is to examine its predictions for various stress or strain histories. Unfortunately it has been difficult to generate stress–strain curves either analytically or numerically with Hart's constitutive law.

Analytic results require the solution of non-linear differential equations. Also, the nearly singular nature of the governing equations for some temperature and strain rate regimes has impeded numerical solutions. This numerical problem has been commonly circumvented by use of an approximation to the full constitutive law termed "the visco-plastic limit" or the "low temperature" approximation[1, 5–7]. Another approach, taken in this paper, is to determine when and how the form of the constitutive law reduces to such a simpler description. In some sense this process is analogous to the construction of a portion of a so-called "deformation map" for the constitutive law.

This paper attempts to elucidate Hart's law by examining uniaxial load histories (with possible unloading but no load reversal) with primary interest on the "visco-plastic" approximation ("visco-plastic limit"). In this approximation, proposed by Hart[1], one of the elements in the model becomes rate independent. The validity of this approximation has been established by the asymptotic results of Hui[8] for the special case of constant inelastic deformation rate. The main result presented here is that the visco-plastic approximation is close to the full constitutive law for a wide range of temperatures and deformation rates. The results we obtain are strictly valid only for small strain uniaxial deformation, though some greater generality is anticipated (but not demonstrated).

The organization of the paper is as follows: Hart's constitutive law for uniaxial deformation, as presented in Hart[1], is described in an introductory manner. The visco-plastic approximation is then defined. An upper bound for the error in using this approximation is found. The specific cases of constant "plastic" state, constant "plastic" strain rate, the relaxation test, and constant inelastic strain rate are then discussed.

No attempt will be made here to justify or critique the constitutive law either from the point of view of its phenomenological relation to experiments or its microscopic rationalization.

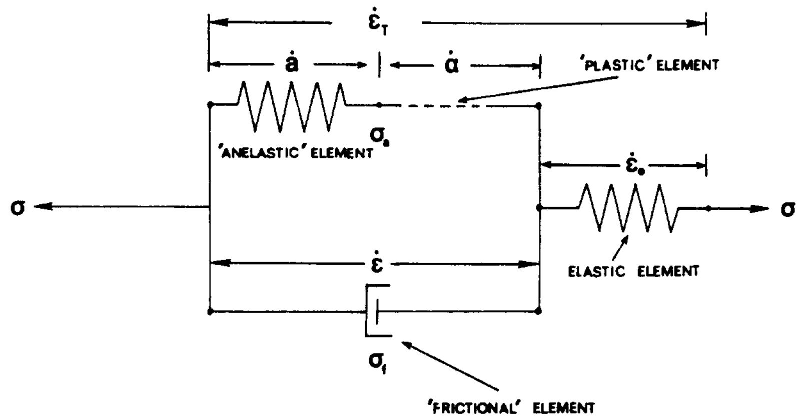

Fig. 1. The assembly of elements in Hart's constitutive model.

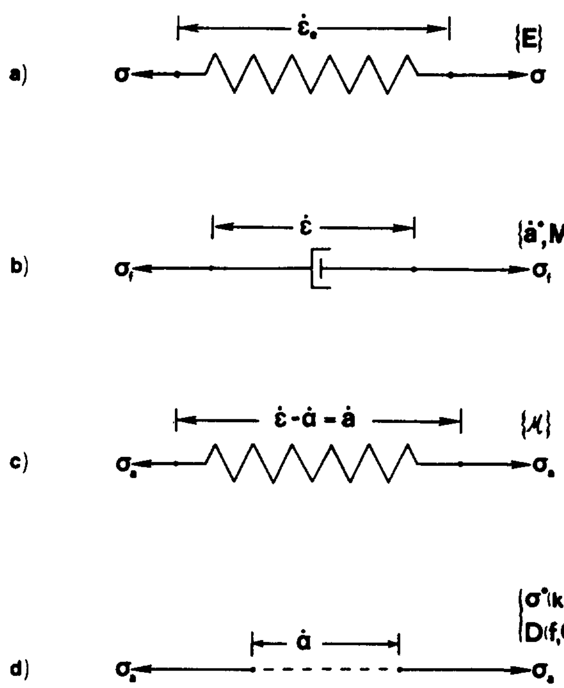

Fig. 2. The four elements in the model. Material constants and state variables used in the definition of the elements are shown in { }'s: (a) the elastic spring, (b) the non-linear "frictional" dashpot, (c) the linear anelastic spring, (d) the plastic element (the evolution of the state variable σ^* depends on k and C and the constant D can be written in terms of f, Q, R and T).

UNIAXIAL CONSTITUTIVE RELATION

The basic constitutive law in uniaxial tension for small deformations, as presented in Hart[1], can be viewed as four simpler elements in combination. This is shown in Fig. 1. The four elements are shown separately in Fig. 2. They are two linear springs, a non-linear dashpot and a special rate-dependent "plastic" element with memory. For problems with finite and/or multidimensional deformation only rate measures of strain $\dot{a}$, $\dot{\varepsilon}$, and $\dot{\alpha}$ are sensibly used in the constitutive law. For conceptual simplicity, however, we will make use of the strain quantities, a, ε, ε_t and α which are only well defined for small monotonic uniaxial deformations. Thus, the results we obtain cannot be rigorously applied to situations with large, multiaxial or reverse deformation histories.

The four basic elements in the constitutive law are described below.

(a) Figure 2(a) shows the linear spring whose deformation rate is the elastic strain rate $\dot{\varepsilon}_e$ and which carries the full true stress σ. E is the usual Young's modulus. This element is in series with the other three elements

$$\dot{\varepsilon}_e = \dot{\sigma}/E. \tag{1}$$

(b) Figure 2(b) shows the non-linear dashpot. The deformation rate of this element is equal to the total inelastic rate $\dot{\varepsilon}$. The deformation rate $\dot{\varepsilon}$ (which is the deformation rate measured relative to the current configuration and thus only equal to $d\varepsilon/dt$ for small deformations) is related to the "frictional" stress σ_f, by the power law relation

$$\dot{\varepsilon} = d\varepsilon/dt = \dot{a}^*(\sigma_f/\mathcal{M})^M \tag{2}$$

where $\dot{a}^*$ is a material constant which also depends on temperature, $\mathcal{M}$ is of the order of the elastic shear modulus G, and M is a material constant of the order of 8–10.

(c) Figure 2(c) shows the second linear spring. The deformation rate in this element is called the anelastic rate and is denoted by $\dot{a}$. It is related to the stress σ_a carried by this element by

$$\dot{\sigma}_a = \mathcal{M}\dot{a} \tag{3}$$

where $\mathcal{M}$ is the same as in the previous element.

(d) Figure 2(d) shows the "plastic" element (also called the $\dot{\alpha}$ element) which also carries the stress σ_a. The deformation rate of this element is $\dot{\alpha}$. This element is of greatest interest in these constitutive equations and has both a rate and history dependence. Before Hart[1] all of the inelastic deformation rate was considered by Hart to be equal to $\dot{\alpha}$ and the two previously mentioned elements (the linear anelastic spring and the non-linear anelastic dashpot) were nonexistent in the model.

The plastic element is defined by the relation between the time histories of σ_a and $\dot{\alpha}$. This is expressed in two steps: the first being the instantaneous relation between σ_a and $\dot{\alpha}$ at fixed plastic state (called "hardness"[1]), and second being the rule for the evolution of the "hardness".

The plastic state ("hardness state") is characterized by the single scalar measure σ^*. The only mechanical manifestation of the state σ^* is through its influence on the relation between σ_a and $\dot{\alpha}$. Curves of σ_a vs $\dot{\alpha}$ at constant state make up a one parameter family, one curve for each value of σ^*.

The following four mutually equivalent expressions state the relation between σ_a and $\dot{\alpha}$ at fixed state σ^*:

$$\dot{\alpha} = D(\sigma^*/G)^m[\ln(\sigma^*/\sigma_a)]^{-1/\lambda} \tag{4a}$$

or

$$\ln(\sigma_a/G) = \ln(\sigma^*/G) - e^{-\lambda[\ln(\dot{\alpha}/D) - m\ln(\sigma^*/G)]} \tag{4b}$$

or

$$\sigma_a = \sigma^* e^{-[(D/\dot{\alpha})(\sigma^*/G)^m]^\lambda} \tag{4c}$$

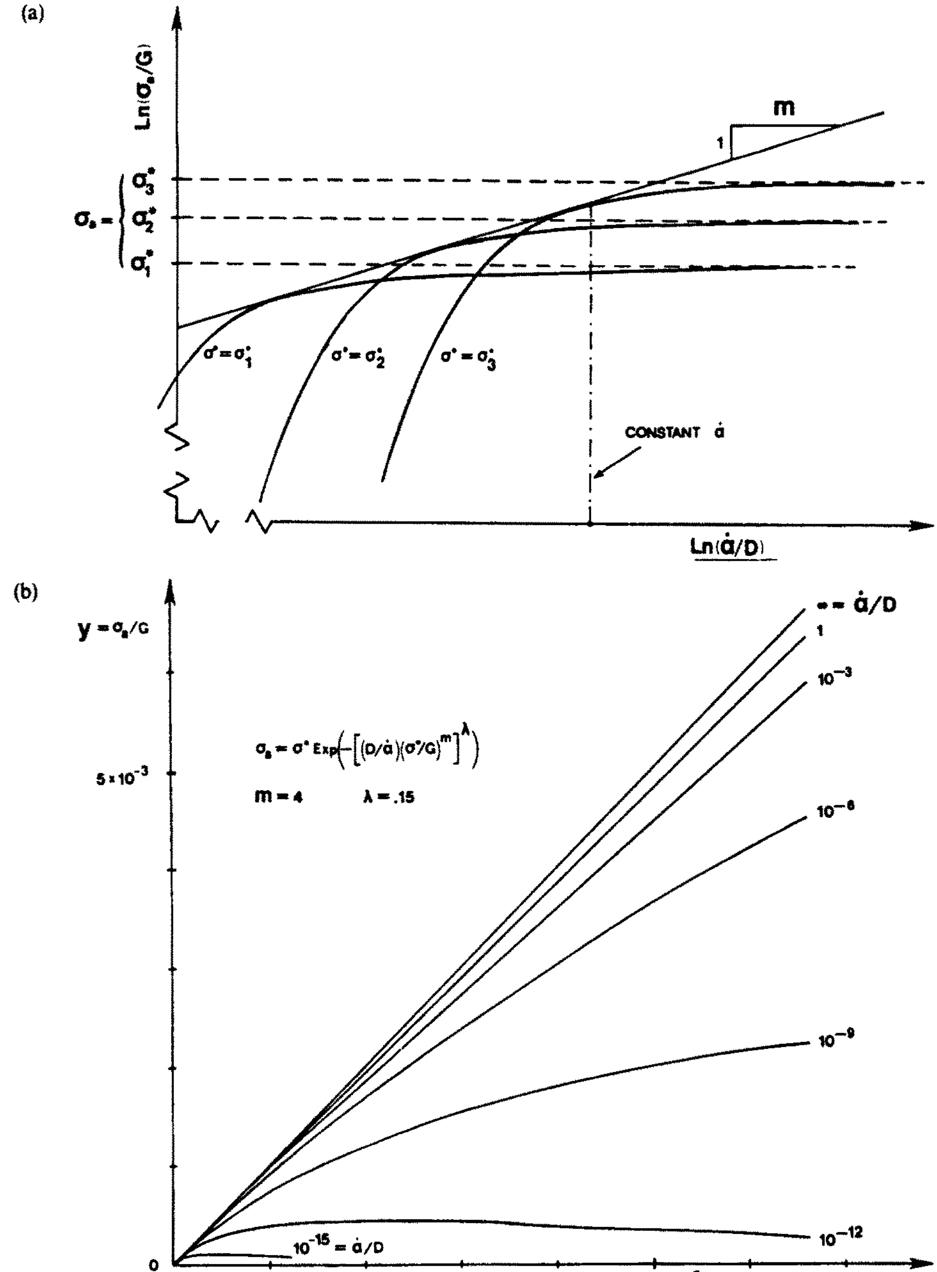

Fig. 3. Constant state relation. (a) "Plastic" stress σ_a vs "plastic" strain rate $\dot{\alpha}$ at a constant state σ^* (log–log). Three constant state curves are shown. (b) σ_a vs σ^* at constant $\dot{\alpha}$.

or

$$\ln(\sigma^*/\sigma_a) = (\dot{\varepsilon}^*/\dot{\alpha})^\lambda \tag{4d}$$

where λ and m are material constants and are said[3] to be of the order of $\lambda \simeq 0.15$ and $m \simeq 4$–5. In these equations D is a temperature-dependent constant given by $D = f\exp(-Q/RT)$ where f and Q are material constants, R is the gas constant and T is the absolute temperature. D is important since it appears in the equations paired with the plastic strain rate $\dot{\alpha}$ and can vary by many orders of magnitude with reasonable temperature variations. Equation (4d), using $\dot{\varepsilon}^* \equiv D(\sigma^*/G)^m$, is the form employed by Hart[1].

Figures 3(a) and (b) show this relation plotted two ways (using $m = 4$, $\lambda = 0.15$). Figure 3(a) plots $\ln(\sigma_a)$ vs $\ln(\dot{\alpha})$ in the manner common for relaxation tests (which are commonly assumed to be constant σ^* tests and in which σ_f is sometimes neglected). The basic features of σ^* can be observed in Fig. 3(a) or eqn (4b) and are claimed[1] to be accurate representations of experimental data.

All of the constant σ^* curves have the same shape (an upside down and backward exponential). Equation (4b) implies that different σ^* constant curves can be obtained from each other by rigid displacement along a straight line with slope $(1/m)$. The saturated value of σ_a at high strain rate is σ^*. Figure 3(b) (showing curves of constant $\dot{\alpha}$) and eqn (4c) also express the equation of constant plastic state. Note that σ^* is strictly greater than σ_a for all plastic strain rates $\dot{\alpha}$. Also, at high $\dot{\alpha}$, σ^* is nearly equal to σ_a for a wide range of σ_a. More precisely, if δ, defined by $\delta \equiv [(D/\dot{\alpha})(\sigma^*/G)^m]^\lambda$, is small then $\sigma^* \simeq \sigma_a$. The parameter δ will appear frequently later in our discussion.

One may observe in Fig. 3(b) or eqn (4c) that the "hardness" has the quality (unexpected from its name) that at a given strain rate the stress σ_a ultimately decreases for large values of "hardness".

In Hart's model, the evolution (rate of change) of σ^* depends on its current value and on the current plastic deformation rate $\dot{\alpha}$. The evolution of the state variable σ^* can also be expressed in terms of the current values of σ_a and σ^* since the constraint eqn (4) (equation of "plastic state") always applies. Any of the following mutually equivalent "hardening" relations may be used:

$$\mathrm{d}(\ln \sigma^*)/\mathrm{d}\alpha = \Gamma, \qquad \mathrm{d}\sigma^* = \sigma^*\Gamma\,\mathrm{d}\alpha, \qquad \dot{\sigma}^* = \sigma^*\Gamma\dot{\alpha} \tag{5a–c}$$

where Γ is a specified function of any two of the three variables σ^*, σ_a and $\dot{\alpha}$. The "hardening" coefficient Γ is not as well constrained by experiment as the plastic state expressions (4). Γ is given in Hart[1] as a representation of the data in Wire *et al.*[9]

$$\Gamma(\sigma^*, \sigma_a) = C(\sigma_a/\sigma^*)^k(\sigma^*/G)^{-m} \tag{6}$$

where C, and k are material constants and m is the same as in eqn (4). Korhonen *et al.*[10] have advocated a different description of Γ than eqn (6). Note for future reference that in eqn (6): (1) Γ is a smooth function of σ_a and σ^* in the vicinity of $\sigma_a = \sigma^*$, and (2) $\Gamma(\sigma^*, \sigma_a)$ is a monotonic increasing function of σ_a for fixed σ^*.

The four elements named above are assembled as in the schematic of Fig. 1. The geometric constraints implied by this figure are that the total deformation rate $\dot{\varepsilon}_t$ is the sum of the inelastic rate $\dot{\varepsilon}$ and the elastic rate $\dot{\varepsilon}_e$. The total inelastic deformation rate is the sum of the anelastic rate $\dot{a}$ and the "plastic" rate $\dot{\alpha}$, i.e.

$$\dot{\varepsilon}_t = \dot{\varepsilon} + \dot{\varepsilon}_e, \qquad \dot{\varepsilon} = \dot{a} + \dot{\alpha}. \tag{7a,b}$$

Similarly, the force balance implied by Fig. 1 shows that the total stress σ is the sum of the "frictional" stress σ_f carried by the non-linear dashpot and the "anelastic" ("plastic") stress σ_a carried by both the anelastic spring and the "plastic" element

$$\sigma = \sigma_f + \sigma_a. \tag{8}$$

Equations (1)–(8) specify Hart's constitutive model in one dimension completely (assuming no load reversal and infinitesimal deformation). To determine a relation between stress and strain, eqns (1)–(8) must be solved simultaneously together with the appropriate initial conditions for the two state variables a, σ^* and information about the deformation history (e.g. $\varepsilon_t(t)$ or $\sigma(t)$). For specified $\sigma(t)$ or $\varepsilon_t(t)$ the equations can be reduced to the solution of two or three coupled first-order differential equations, respectively.

THE VISCO-PLASTIC APPROXIMATION

Hart's model, as presented above, is conceptually simple except for the behavior of the plastic element. However, this element has, in a few calculations[5, 7] been replaced by a simpler rate independent element. This substitution, to be described below, is termed the "visco-plastic" approximation. In the "visco-plastic" approximation, first discussed by Hart[1], the inelastic constitutive law reduces to a (non-linear) viscous element in parallel with the series combination of a rate-independent plastic element and a linear spring.

The approximation, usually motivated for computational reasons, has not been rigorously justified. In the following sections we will rationalize the use of the visco-plastic approximation for some time, temperature and stress regimes.

The visco-plastic approximation in this paper, like that in Ref. [7], is somewhat stricter in interpretation than that which has been used in some numerical work. We interpret the visco-plastic approximation as a substitute constitutive law to be used throughout a deformation history. This is in contrast to the visco-plastic "limit" of only using the approximation at particular points in the history where the nearly singular behavior of the full constitutive law causes numerical problems.

The visco-plastic approximation is defined by the following replacements for eqns (4) and (5) (eqn (6) or its equivalent, is maintained):

$$\begin{aligned} &\sigma_a = \sigma^* \quad \text{and} \quad d\sigma^*/d\alpha = \sigma^*\Gamma(\sigma^*, \sigma_a = \sigma^*) && \text{if } \dot\alpha > 0 \\ &\dot\sigma^* = 0 && \text{if } \dot\alpha = 0 \\ &\dot\alpha \ = 0 && \text{if } \sigma_a < \sigma^*. \end{aligned} \tag{9}$$

The visco-plastic approximation can be expressed in integral form as

$$\alpha_{vp} = \int_{\sigma_0^*}^{(\sigma_a)_{max}} [\eta\Gamma(\eta, \eta)]^{-1}\, d\eta \qquad \text{if } (\sigma_a)_{max} \geqslant \sigma_0^* \tag{10a}$$

$$\alpha_{vp} = 0 \qquad \text{if } (\sigma_a)_{max} \leqslant \sigma_0^* \tag{10b}$$

where $(\sigma_a)_{max}$ is the maximum value of σ_a that has occurred in the loading history, σ_0^* is the initial value of σ^* and α_{vp} denotes the accumulated plastic strain (i.e. $\int\dot\alpha\, dt$) using the visco-plastic approximation.

We have set $\alpha = 0$ at the start of the deformation history ($t = 0$) for simplicity. Equations (9) and (10) describe classical, rate-independent, work hardening, plasticity. Using the form for Γ in eqn (6)[1] we obtain by evaluating the integral in eqn (10a) (and assuming $\alpha_{vp} > 0$)

$$\alpha_{vp} = (1/mC)\{[(\sigma_a)_{max}/G]^m - [\sigma_0^*/G]^m\}. \tag{11}$$

The work hardening is by a power law in this case. Thus, within the visco-plastic approximation, the "plastic" element in Hart's model is replaced with a more conventional plasticity element. If $\dot\alpha$ is always positive we may replace the "plastic" element with a deformation element where σ_a only depends on the current value of α.

Perfect plasticity in the "plastic" element is obtained by the additional limit $C \to 0$.

MOTIVATION FOR THE VISCO-PLASTIC APPROXIMATION

In Hart's[1] original brief description the limiting behavior defined by eqn (9), (10), or (11) is obtained by the limit $\lambda \to 0$. This is apparently a typographical error since the limit does not lead to the desired approximation.

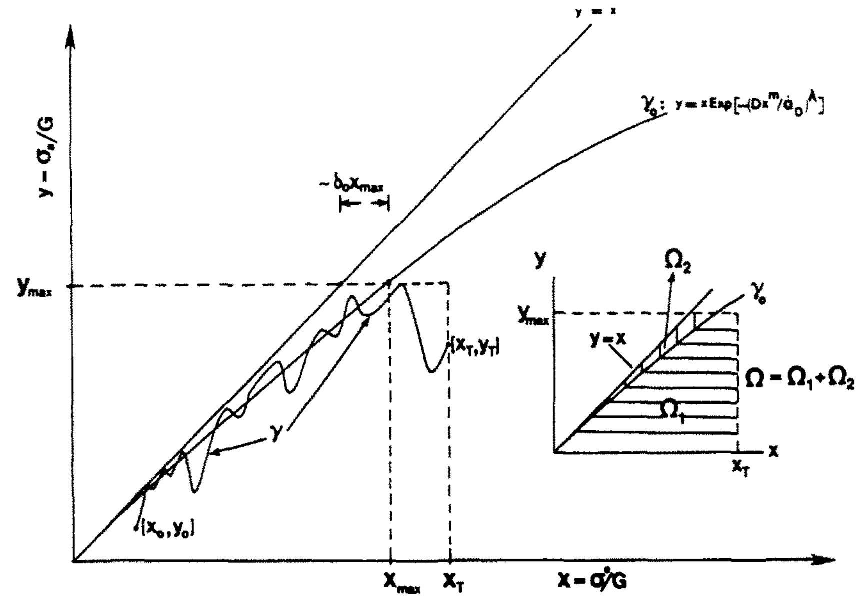

Fig. 4. Stress-state trajectory γ during general deformation. Regions and points used in the analysis are marked.

On the other hand, as argued in Van Arsdale *et al.*[7], if D is very close to 0 (low temperature) then, for a wide range of strain rates and stress, σ_a is approximately equal to σ^* by eqn (4c). Thus, for non-negligible $\dot{\alpha}$, Γ in eqn (6) can be evaluated with great accuracy using $\sigma_a = \sigma^*$. By eqn (5), σ^* only evolves when $\dot{\alpha}$ is non-negligible. This roughly motivates use of the visco-plastic approximation, eqn (9) or eqn (10).

Note that if D is not exactly zero then setting $\sigma_a = \sigma^*$ is in no way an approximate solution to eqn (4a). This is because eqn (4a) becomes singular for $\sigma_a = \sigma^*$. In what sense, then is the visco-plastic approximation a good approximation for finite values of D?

ERROR BOUND FOR THE VISCO-PLASTIC APPROXIMATION

In this section we establish an approximate upper bound, $(\Delta\alpha)_{min}$, for the difference between the accumulated plastic strain predicted by the full constitutive law and that predicted by the visco-plastic approximation in eqns (22c) and (23b). The error bound is expressed in terms of the maximum value of σ_a in the deformation history and the total time t_T of the history, or, more approximately in terms of t_T and α_{vp0} (the plastic strain that the visco-plastic approximation predicts for a special comparison test). The argument which follows is this: both a lower bound and an upper bound for the accumulated plastic strain are found and related to the strain that the visco-plastic model would have computed. The upper bound is expressed in terms of a parameter. The parameter is then chosen so as to minimize the difference between the two bounds. An error estimate is thus obtained. The details that follow are straightforward but somewhat involved.

For notational convenience we define the following variables which are used with reference to Fig. 4. Let $x = \sigma^*/G$ and $y = \sigma_a/G$, i.e. the stress quantities are normalized by the shear modulus G. Let $y(t)$ $(0 < t < t_T)$ denote an arbitrary stress history, where t_T denotes the duration of the stress history. Let $x(t)$ be the corresponding history of "hardness" using the full constitutive law. Let y_0 and x_0 denote the initial values of σ_a/G and σ^*/G at time $t = 0$. The final value of x and y at time t_T are denoted by x_T and y_T, respectively. The maximum value of $y(t)$ in the time interval $[0, t_T]$ is y_{max}. The stress and hardness history of the plastic element consistent with the full constitutive law can be represented by the curve $\gamma = (x(t), y(t))$. Note that $\dot{\sigma}^* > 0$ implies that $x(t)$ is a monotonic function of t. Therefore, y is a single valued function of x on γ. The curve γ lies completely in the region Ω bounded by the x-axis, the line $x = x_T$, the line $y = y_{max}$ and the line $y = x$. Let γ_0 be a constant $\dot{\alpha}$ curve for fixed D (i.e. fixed temperature) defined by

$$y = x\exp\{-[(D/\dot{\alpha}_0)x^m]^\lambda\}$$

where $\dot{\alpha}_0$ is a specified number whose value is to be assigned later in the argument. The maximum of y on γ_0 is attained when $[(D/\dot{\alpha}_0)x^m]^\lambda = 1/m\lambda$, which is of order unity and occurs to the right of the domain shown in Fig. 4. We define $\delta_0 = [(D/\dot{\alpha}_0)(x_{max})^m]^\lambda$ where x_{max} is defined as the value of x corresponding to y_{max} on γ_0. Note that $x_{max} > y_{max}$. We assume, as must be checked when $\dot{\alpha}_0$ is finally assigned a value, that y_{max} and $\dot{\alpha}_0$ are such that $\delta_0 \ll 1$. With this assumption the curve γ_0 is monotonic and is accurately described by

$$y = x(1 - \delta_0). \tag{12}$$

The assumption $\delta_0 \ll 1$ implies that γ_0 divides Ω into two non-intersecting regions Ω_1 and Ω_2, where Ω_2 is the region between the line $y = x$, the curve γ_0 and the line $y = y_{max}$ as shown in Fig. 4.

The following assumptions regarding the function $\Gamma(\sigma^*, \sigma_a)$ or, with a slight abuse of notation, $\Gamma(x, y)$, are used.

(a) $\Gamma(x, y)$ is continuously differentiable in Ω_2 and $\Gamma(x, x) > 0$.

(b) $y_1 > y_2$ implies $\Gamma(x, y_1) > \Gamma(x, y_2)$ for all $x > 0$. These conditions are satisfied by eqn (6).

We now establish a lower bound for the accumulated plastic strain α. The following observations are used: $\sigma_a(t) < \sigma^*(t)$ at all points in a loading history and hence $y < x$ on γ. $\Gamma(x, y)$ is a monotonic increasing function of y so that $1/\Gamma(x, y) > 1/\Gamma(x, x)$. This implies

$$\begin{aligned}\alpha &= \int_{x_0}^{x_T} [\eta\Gamma(\eta, y(\eta))]^{-1}\,d\eta \\ &> \int_{x_0}^{x_T} [\eta\Gamma(\eta, \eta)]^{-1}\,d\eta \\ &> \int_{x_0}^{y_{max}} [\eta\Gamma(\eta, \eta)]^{-1}\,d\eta = \alpha_{vp}\end{aligned} \tag{13}$$

assuming $y_{max} \geqslant x_0$. If $y_{max} < x_0$ then $\alpha_{vp} = 0$ by eqn (10b). Thus, the accumulated plastic strain predicted by the visco-plastic approximation, α_{vp}, is a lower bound for α.

We will now establish an upper bound for the accumulated plastic strain α during the load history γ. The total plastic strain α accumulated during the deformation history γ is equal to the sum of α_1 and α_2, where α_1 and α_2 are the plastic strain accumulated in Ω_1 and Ω_2, respectively. An upper bound for the accumulated plastic strain in region Ω_1 is

$$\alpha_1 \leqslant \dot{\alpha}_0 t_T \tag{14}$$

since $\dot{\alpha} \leqslant \dot{\alpha}_0$ in Ω_1.

We next find an upper bound for α_2 in Ω_2 assuming $y_{max} \geqslant x_0$. The following identity is an immediate consequence of condition (a) above and eqn (4c):

$$\Gamma(x, y) = \Gamma(x, x\,e^{-\delta}) = \Gamma(x, x) - x\delta\partial\Gamma/\partial y|_{y=x} + O(\delta^2) \tag{15a}$$

where $\delta = [(D/\dot{\alpha})x^m]$. Equation (15a) above and $\delta < \delta_0$ in Ω_2 implies that

$$\Gamma(x, y) \geqslant \Gamma(x, x) - x\delta_0 \partial\Gamma/\partial y\big|_{y=x} \qquad \text{in } \Omega_2 \tag{15b}$$

to first order in δ. All inequalities from this point on are correct at least to first order in δ. Generalization to higher order does not seem sufficiently rewarding to warrant the added complexity in the argument. If we define

$$h(x) \equiv \Gamma(x, x) \qquad \text{and} \qquad h_1(x) \equiv \partial\Gamma/\partial y\big|_{y=x} \tag{16}$$

then by eqns (15b) and (5c)

$$\dot{\alpha} = \dot{x}/x\Gamma(x, y) \leqslant \dot{x}\{xh(x)[1 - (h_1(x)/h(x))x\delta_0]\}^{-1} \tag{17}$$

or

$$\dot{\alpha} \leqslant (\dot{x}/xh(x))[1 + \delta_0 x h_1(x)/h(x)] \tag{18}$$

in Ω_2. We now define a function $P(x)$ which allows us to keep track of when γ is in Ω_2. $P(x)$ takes on the values 0 or 1 depending on γ as follows:

$$\begin{aligned} P(x) &= 1 \quad \text{if } (x, y) \text{ on } \gamma \text{ is in } \Omega_2 \\ P(x) &= 0 \quad \text{otherwise.} \end{aligned}$$

Inequality (18) above implies that

$$\alpha_2 \leqslant \int_{x_0}^{x_{max}} [P(\eta)/\eta h(\eta)]\, d\eta + \delta_0 \int_{x_0}^{x_{max}} [P(\eta) h_1(\eta)/h^2(\eta)]\, d\eta \tag{19a}$$

$$\alpha_2 \leqslant \alpha_{vp} + \delta_0 \left\{ 1/h(y_{max}) + \int_{x_0}^{y_{max}} [h_1(\eta)/h^2(\eta)]\, d\eta \right\}. \tag{19b}$$

In deriving the above inequalities, we have made use of the facts that $h(x) = \Gamma(x, x) > 0$, $\Gamma(x, y)$ is monotonically increasing in y, $P(x) \leqslant 1$, $h(x)$ is smooth and that $x_{max} \simeq (1 + \delta_0)y_{max}$. If the test starts with a high value of σ^* so that $x_0 > y_{max}$ then the visco-plastic approximation predicts that $\alpha = \alpha_{vp} = 0$. Equations (19) above may then be made sensible by using the value x_0 for y_{max}. This substitution is justified because it is equivalent to replacing the stress history with one that has greater stress (and hence greater accumulated α) at every instant in time.

Inequalities (13), (14) and (19b) bound the accumulated plastic strain α from above and below

$$\alpha_{vp} \leqslant \alpha \leqslant \alpha_{vp} + \Delta\alpha \tag{20a}$$

with

$$\Delta\alpha = \dot{\alpha}_0 t_T + B\delta_0 \tag{20b}$$

and

$$B = 1/h(y_{max}) + \int_{x_0}^{y_{max}} h_1(\eta)/h^2(\eta)\,d\eta \tag{20c}$$

where $\Delta\alpha$ is an error estimate and B depends only on properties of Γ.

The value of $\dot{\alpha}_0$ is now chosen so that $\Delta\alpha$ is minimized. The minimum of $\Delta\alpha$, $(\Delta\alpha)_{min}$, is obtained by solving $d(\Delta\alpha)/d\dot{\alpha}_0 = 0$ for $\dot{\alpha}_0$. Carrying out this procedure (using $\delta_0 = [(D/\dot{\alpha}_0)y_{max}^m]^\lambda$ to first order) yields

$$(\dot{\alpha}_0)_{min} = [\lambda(Dy_{max}^m)^\lambda B/t_T]^{1/(1+\lambda)} \tag{21a}$$

$$(\delta_0)_{min} = [Dy_{max}^m t_T/B\lambda]^{\lambda/(1+\lambda)} \tag{21b}$$

$$(\Delta\alpha)_{min} = B(\delta_0)_{min}F(\lambda) \tag{21c}$$

where $F(\lambda) = [\lambda^{1+\lambda} + 1]^{1/(1+\lambda)}$. $F(\lambda)$ has the value of 1.1 when $\lambda = 0.15$ and may be set to 1 for practical purposes. B is defined in eqns (20c) and (16) and is determined by the form of the hardening coefficient Γ. $(\Delta\alpha)_{min}$ is our best estimate of the error in plastic strain by using the visco-plastic approximation.

For the error estimate to be valid one needs to check that $(\delta_0)_{min} \ll 1$ since this condition was assumed throughout the derivation. Comparison of eqns (21c) and (20b) shows that the γ_0 curve which minimizes the error estimate causes most of the error estimate to come from Ω_2, i.e. near the curve $y = x$.

$(\Delta\alpha)_{min}$ can be evaluated if the hardening coefficient is given by eqn (6)

$$h(\eta) = C\eta^{-m}, \qquad h_1(\eta) = kh(\eta)/\eta, \qquad B = \beta y_{max}^m/C \tag{22a}$$

$$(\delta_0)_{min} = [DCt_T/\beta\lambda]^{\lambda/(1+\lambda)}F(\lambda) \tag{22b}$$

$$(\Delta\alpha)_{min} = (\beta y_{max}^m/C)[DCt_T/\beta\lambda]^{\lambda/(1+\lambda)}F(\lambda) \tag{22c}$$

where $\beta \equiv 1 + (k/m)[1 - (x_0/y_{max})^m]$ ranges between 1 (if $x_0 = y_{max}$ as in an ideal relaxation test) and $1 + k/m$ (if $x_0 = 0$ as with a totally unhardened material). Dropping all terms of order 1 (i.e. β, $F(\lambda)$, $\lambda^{-\lambda/(1+\lambda)}$) the error estimate becomes

$$(\Delta\alpha)_{min} \simeq (y_{max}^m/C)(DCt_T)^{\lambda/(1+\lambda)} \tag{23a}$$

$$= m\alpha_{vp0}(DCt_T)^{\lambda/(1+\lambda)} \tag{23b}$$

where α_{vp0} represents the strain predicted by the visco-plastic approximation for the given peak stress y_{max} but with $x_0 = 0$ (whether or not $x_0 = 0$ in the actual load history).

The error bound developed in this section (eqns (21c), (22c) or (23b)) is crude in that it only uses the peak stress y_{max} and the duration of the load history t_T. The bound is for the maximum error for every history described by these two numbers. It is possible that the visco-plastic approximation is much better than the bound for particular sections of some load histories. Note that the bound is expressed in terms of the stress σ_a and not the total stress σ. For $\dot{\varepsilon} > 0$, however, $\sigma > \sigma_a$ and the bound is still applicable if the peak value of the total stress σ is used in the calculation.

In some applications it may be of interest to estimate the error in stress when strain is controlled. The error in σ_a for given α can be estimated by calculating σ_a with the visco-plastic approximation (eqn (11)) from α. The error in σ_a is then roughly given by $(d\sigma_a/d\alpha_{vp})\Delta\alpha$.

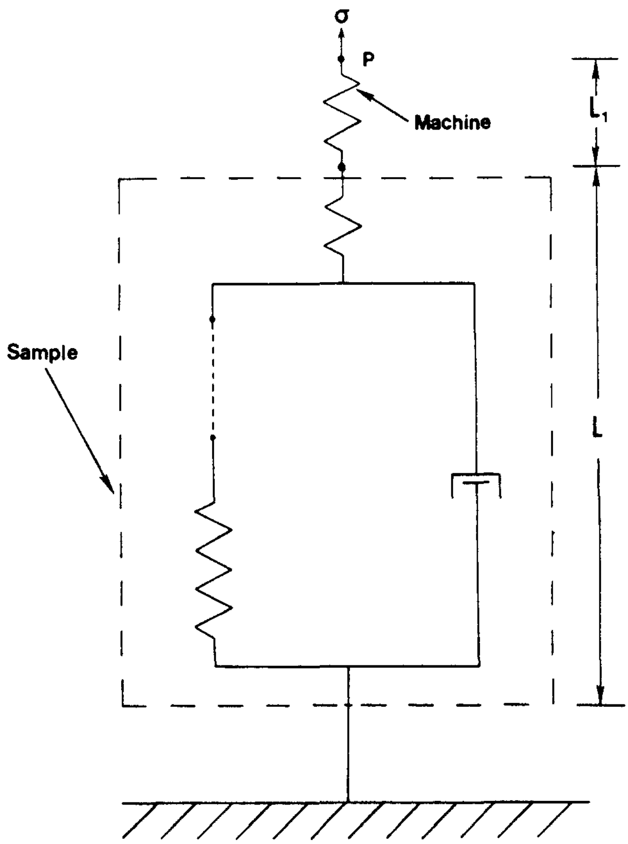

Fig. 5. Schematic diagram of a material test. In "load relaxation" the load point P is held fixed after extension.

ANALYSIS OF SOME SIMPLE DEFORMATION HISTORIES

A few deformation histories which are simple to analyze are considered in the following sections. Some have little direct relevance to actual material tests. We use them here because they have simple governing equations.

Constant plastic strain rate, $\dot{\alpha}$ = const.

In a constant plastic strain rate test, σ_a is a single-valued function of σ^* as shown in the constant $\dot{\alpha}$ curve in Fig. 3(b). At low temperature or very high plastic strain rate, i.e. $[(D/\dot{\alpha})(\sigma^*/G)^m]^\lambda = \delta \ll 1$, we have that $\sigma_a \simeq \sigma^*$. For sufficiently high temperatures or very large values of σ^* the condition $\delta \ll 1$ is no longer satisfied and σ_a starts to deviate from $\sigma_a \simeq \sigma^*$.

The relaxation test

The load relaxation test is used by Hart and co-workers for experimental verification of the constitutive model[7, 9, 11]. The relaxation test has been used because it is assumed to be a constant state test for practical purposes (i.e. σ^* remains constant for the duration of the test) and thus reveals the constant state relation eqn (4) as well as the current value of the state variable σ^*. Of course $\dot{\sigma}^* > 0$ during the test (by eqns (4)–(6)). The implicit assumption is that the accumulated change in σ^* is small compared to the change in σ_a.

The load relaxation test is shown schematically in Fig. 5. The specimen is allowed to undergo some loading history (e.g. a gradual upward movement of the load point P in Fig. 5), the load point is then fixed for the duration of the test. In the following analysis, $t = 0$ corresponds to the initiation of the relaxation test.

Geometric constraint (load point fixed for $t > 0$) implies that

$$\dot{L}_1 + \dot{L} = 0.$$

Assuming no inelastic deformation of the machine, this implies

$$\dot{\varepsilon}_t + \dot{\sigma}/K_m = 0$$

where σ is the stress carried by the sample, K_m the effective stiffness of the loading machine (which is the actual stiffness times the ratio of the sample length to its cross-sectional area), and ε_t is the total strain in the sample.

Equations (1), (3), (7), and (8) then imply that in a relaxation test

$$\dot{\alpha} = -\dot{\sigma}_a(1 + \dot{\sigma}_f/\dot{\sigma}_a)\{1/K_m + 1/E + [(1/\mathscr{M})(1 + \dot{\sigma}_f/\dot{\sigma}_a)]\}. \tag{24}$$

The full governing differential equations for $\sigma(t)$ in the relaxation test can be found by combining eqn (24) with eqns (2) and (4)–(6). We will only consider the case when $\dot{\sigma}_f/\dot{\sigma}_a \ll 1$. This assumption is often made, either implicitly or explicitly in description and usage of the relaxation test as applied to Hart's model. The evolution equations of σ^* can be written using eqn (24) as

$$\dot{\sigma}^* = -\sigma^*\Gamma(\sigma^*, \sigma_a)[(1/K_m + 1/E)(1 + \dot{\sigma}_f/\dot{\sigma}_a) + 1/\mathscr{M}]\dot{\sigma}_a. \tag{25}$$

The assumption $\dot{\sigma}_f/\dot{\sigma}_a \ll 1$ implies approximately

$$\dot{\sigma}^* = -\sigma^*\Gamma(\sigma^*, \sigma_a)\dot{\sigma}_a/K_T \tag{26a}$$

or

$$\mathrm{d}\sigma^*/\mathrm{d}\sigma_a = -\sigma^*\Gamma(\sigma^*, \sigma_a)/K_T \tag{26b}$$

where $1/K_T = [1/K_m + 1/E + 1/\mathscr{M}]$. If the hardening coefficient is given by eqn (6), eqn (26b) can be solved as

$$x = x_0\{[(m + k)/(k + 1)](CGx_0^{1-m}/K_T)[1 - (y/x_0)^{k+1}] + 1\}^{1/(m+k)} \tag{27}$$

where $x = \sigma^*/G$ and $y = \sigma_a/G$. The integration constant x_0 is the value of x and y when $y = x$ and is presumably roughly the beginning of a relaxation test.

Two quantities of interest can be determined immediately. $\mathrm{d}x/\mathrm{d}y$ at x_0, the relative rate of change of σ^* to that of σ_a is by eqn (26b)

$$\mathrm{d}\sigma^*/\mathrm{d}\sigma_a|_{\sigma^* = \sigma_a = \sigma_0^*} = \mathrm{d}x/\mathrm{d}y|_{y = x = x_0} = -x_0\Gamma(x_0, x_0)G/K_T = -CGx_0^{1-m}/K_T. \tag{28}$$

Also, the total accumulation of σ^* in an infinitely long test (where $y \to 0$) is from eqn (27)

$$\Delta x \simeq CGx_0^{-m}/[(k + 1)K_T] \tag{29}$$

assuming $\Delta x \ll x_0$. Our analysis depends on the assumption $\dot{\sigma}_f/\dot{\sigma}_a \ll 1$ during the relaxation test. Using $\sigma_f = M[(1/K_m + 1/E)(-\dot{\sigma})]^{1/M}$ in the relaxation test this condition is equivalent to the following consistency condition:

$$\mathscr{M}[(1/K_m + 1/E)/\dot{a}^*]^{1/M}(-\dot{\sigma})^{(1/M)-2}(\mathrm{d}^2\sigma/\mathrm{d}t^2)/M \ll 1. \tag{30}$$

That is, if eqn (30) is not satisfied then $\dot{\sigma}_f$ cannot be neglected in the analysis as in the paragraphs above.

Continuing with the assumption that $\dot{\sigma}_f$ is negligible, the condition of constant state ($\dot{\sigma}_f \ll \dot{\sigma}_a$) and eqn (28) imply that the relaxation test is a constant state test only if the condition

$$CGx_0^{1-m}/K_T \ll 1 \tag{31}$$

is satisfied. For the limiting case of an infinitely stiff machine, where $K_T^{-1} = 1/\mathcal{M} + 1/E \simeq 1/G$, eqn (31) becomes

$$Cx_0^{1-m} \ll 1. \tag{32}$$

Equation (31) restricts conditions for which the relaxation test is a constant state test. From eqns (9) and (6) the visco-plastic approximation is $dx/d\alpha = Cx^{1-m}$; the term Cx^{1-m} is the usual tangent modulus (normalized by G). Therefore, the relaxation test can be approximated by a constant state test if the tangent modulus of the "plastic" element is small compared to the effective elastic stiffness of the machine–sample combination. If eqn (32) is not satisfied there is no machine for which the constant state assumption is justified.

Constant inelastic strain rate

The governing equations for the evolution of σ_a and σ^* in a constant $\dot{\varepsilon}$ test can be obtained using eqns (4a), (5a), (6) and (7a). They are

$$d\sigma_a/dt = \mathcal{M}[\dot{\varepsilon} - D(\sigma^*/G)^m[\ln(\sigma^*/\sigma_a)]^{-1/\lambda} \tag{33a}$$

$$d\sigma^*/dt = DC\sigma^*(\sigma_a/\sigma^*)^k[\ln(\sigma^*/\sigma_a)]^{-1/\lambda}]. \tag{33b}$$

Note that, for a given inelastic strain rate, σ_f is completely determined by eqn (2) so σ_f is decoupled from the equations governing σ_a and σ^*. Eliminating t in eqn (33), we obtain, in terms of the dimensionless quantities $x = \sigma^*/G$ and $y = \sigma_a/G$

$$\begin{aligned} dy/dx &= \{1 - \delta_1 x^m[\ln(x/y)]^{-1/\lambda}\}\{\delta_2 x(y/x)^k[\ln(x/y)]^{-1/\lambda}\}^{-1} \\ &= f(x, y) \end{aligned} \tag{34a}$$

with

$$\delta_1 = D/\dot{\varepsilon}, \qquad \delta_2 = GDC/\mathcal{M}\dot{\varepsilon}.$$

The initial condition for eqn (34a) is

$$y(x = x_0) = y_0 \tag{34b}$$

where x_0 is the initial dimensionless hardness and y_0 the initial dimensionless σ_a.

The behavior of eqn (34a) for small values of δ_1 and δ_2 will now be examined on the phase plane. This problem has been investigated by Hui[8] using matched asymptotics (note: Hui[8] defines x and y differently). A typical point on the phase plane is denoted by (x, y). The region of interest in the phase plane is $x > 0$, $y > 0$, $x > y$. Only part of this region is of practical interest since the hardness σ^* is always expected to be much smaller than G, i.e. $x \ll 1$.

Define the isocline $y = g(x, b)$ by the set of points where solutions to eqn (34) have slope b. The function $g(x, b)$ is thus defined implicitly by

$$f(x, g(x, b)) = b \tag{35}$$

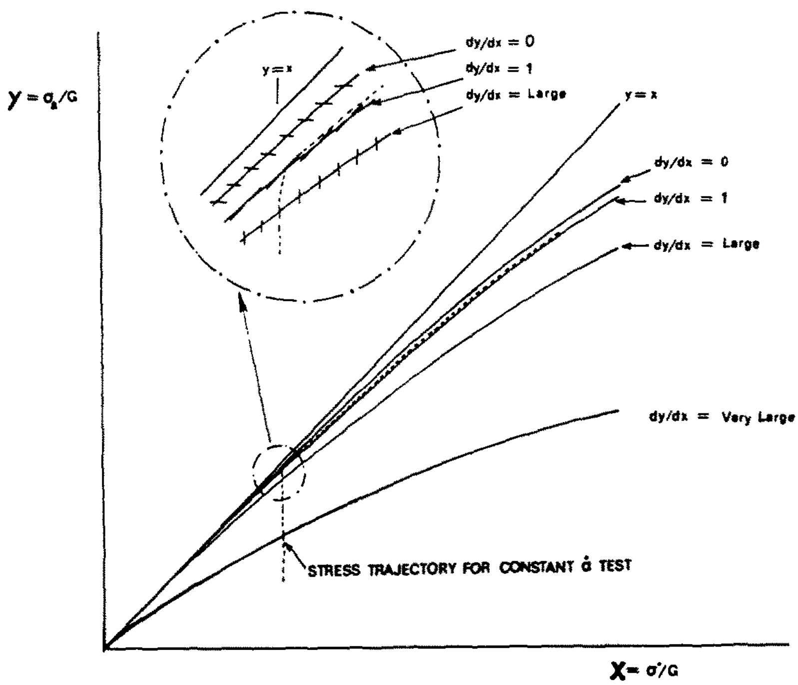

Fig. 6. Phase plane for constant $\dot{\varepsilon}$ test. Isoclines of constant dy/dx are marked as well as a possible stress trajectory.

where $f(x, y)$ is defined in eqn (34). Several isoclines are shown schematically in Fig. 6. Except when $b = 0$, one cannot find a closed form solution for the isocline. However, it is possible to show analytically the following properties.

(a) For any $b \geqslant 0$, there exists a unique curve $y = g(x, b)$. Furthermore, the isoclines are non-intersecting (except at the origin) with $g(x, b_1) > g(x, b_2)$ if $b_2 > b_1$.

(b) For any given $b \geqslant 0$, the vertical distance from a point y lying on the curve $y = g(x, b)$ to the line $y = x$, i.e. $x - g(x, b)$, is a monotonic increasing function of x for any fixed b. The slope of each curve is less than one for all $x > 0$.

(c) The asymptotic behavior of this family of curves for small values of x is

$$y = x(1 - b\delta_2 x^{\lambda}) \quad \text{for } b > 0, \quad x \to 0. \tag{36}$$

The isocline for $b = 0$, in which case $\dot{\varepsilon} = \dot{\alpha}$, is given by

$$y = x\{\exp[(-\delta_1 x^m)^{\lambda}]\} \qquad \text{for all } x.$$

Clearly a solution curve crosses an isocline $g(x, b)$ from below if and only if the slope of the solution curve is more than the slope of the isocline at the point of intersection

$$f(x, g(x, b)) > \mathrm{d}y(x, b)/\mathrm{d}x.$$

Two isoclines are of particular interest: the curve $g(x, b = 0)$ and the curve $g(x, b = 1)$. Under conditions of low temperature, the parameters δ_1 and δ_2 are extremely small. This means that the maximum of the curve $y = g(x, 0)$ occurs at $x(0) \gg 1$ (or $\sigma_* \gg G$). Small values of δ_1 and δ_2 imply that the curves $y = g(x, 1)$ and $y = g(x, 0)$ are close to each other and also close to the line $y = x$ in the region where $x \ll 1$.

The solution of the initial valued problem $y(x_0) = y_0$ with x_0 and $y_0 \ll 1$, must cross the curve $g(x, 1)$ but cannot cross the curve $y = g(x, 0)$. The solution, squeezed between these two curves, must therefore be well approximated by either one of them. It is

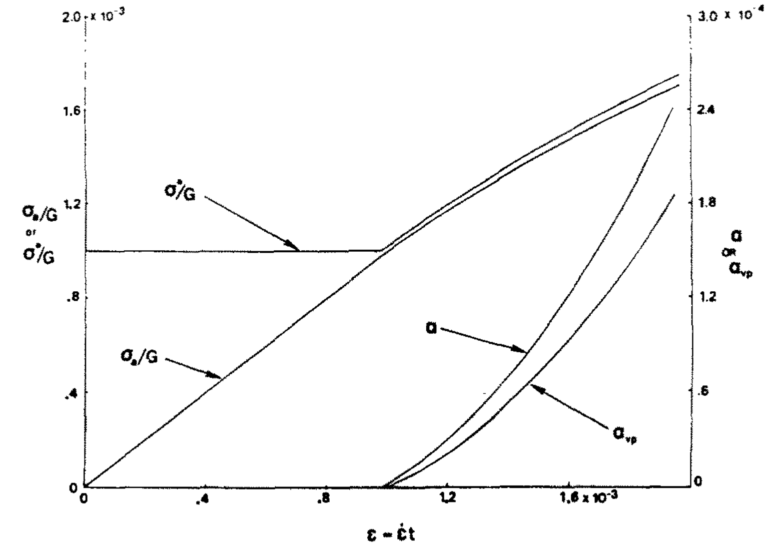

Fig. 7. Simulation of constant inelastic strain rate test. σ_a, σ^*, α are determined by numerical integration of eqns (33) and (5). α_{vp} is calculated from σ_a using eqn (11). Parameters (listed in the text) are chosen so that $\alpha \not\approx \alpha_{vp}$ but the error bound is still applicable.

anticipated that the solution should be much closer to the curve $g(x, 1)$ than to the curve $y = g(x, 0)$ since the isoclines themselves have slope very close to unity. This is supported by the asymptotic analysis of Hui[8]. The above analysis is shown schematically in Fig. 6.

The argument just presented (in fact the original motivation for Hui[8] and the more general error estimate presented previously), shows why the visco-plastic approximation works at low temperature for constant inelastic strain rate. Solution curves $y(x)$ have very large slope for y not near x and are constrained very near the curve $y = x$ once y is sufficiently large.

Numerical example: constant inelastic strain rate

We use the case of constant inelastic strain rate $\dot{\varepsilon}$ to test and illustrate the error estimates we have made. The three differential equations, eqns (5c), (33a), and (33b) (using eqn (6) in eqn (5c)) are numerically integrated. Plots of α, σ^*, σ_a vs strain ε (which is also normalized time for this test) are shown in Fig. 7. Also plotted is α_{vp}, the visco-plastic strain predicted by eqn (11) using σ_a.

Parameter values used in this simulation are: $m = 4$, $\lambda = 0.15$, $k = 7$, $D/\dot{\varepsilon} = 1$, $C = 1 \times 10^{-8}$, $\mathcal{M} = G$, $\sigma^*/G = 10^{-3}$ at $t = 0$, $\sigma_a = \alpha = 0$ at $t = 0$.

The values of $D/\dot{\varepsilon}$ and C correspond to deformation at fairly high temperature or low strain rate where the visco-plastic approximation is not expected to be accurate. Though there is no visible change in either α or σ^* before σ_a reaches σ^*, the curves for α and α_{vp} are visibly separate as are the curves for σ_a and σ^*. For much smaller $D/\dot{\varepsilon}$ the α and α_{vp} curves would be indistinguishable as would be the σ_a and σ^* curves.

At the end of the simulation, when $\sigma_a/G = 1.73 \times 10^{-3}$, the numerically predicted value for α is 2.67×10^{-4} which is greater than α_{vp} by 0.66×10^{-4}. The error bound estimate eqn (22c) predicts $\Delta\alpha_{min} = 1.15 \times 10^{-4}$. Thus the error in using the visco-plastic approximation is a little over half of that predicted by the error bound estimate. The rough error bound in eqn (23b) slightly underestimates the error at 0.36×10^{-4}.

CONCLUSION

We have reviewed Hart's law for inelastic deformation in the simplest possible form: small strain, uniaxial deformation with positive load. The basic nature of the law is somewhat clarified by the realization that it is well approximated by a simple combination of classical components at low temperature: two linear springs, a non-linear dashpot and a work hardening rate-independent plastic element.

An error bound for the visco-plastic approximation (eqns (21c), (22c), and (23b)) for a fairly general class of load histories, has been found in terms of the peak load and the total time of the test.

The results presented do not address the accuracy of the visco-plastic approximation at every instant in time for rate quantities. In fact, the approximation may be very good for stress and accumulated deformation while poor for instantaneous rates during some parts of the deformation history.

One could presumably use the bounds presented here to determine whether the visco-plastic approximation could replace the full constitutive law in numerical simulation. Alternatively, for a given desired accuracy and loading conditions there exists a range of values of D for which the full constitutive law is tolerably approximated by the visco-plastic approximation. If the value of D describing the material of interest is in this range then, in the numerical computation, one can use the largest D in the range without compromising on accuracy. This procedure may reduce the stiffness of the governing equations enough so that it is unnecessary to explicitly use the visco-plastic approximation in the numerical integration.

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